Automatic Control

Chapter three

Mathematical Modeling of mechanical end electrical systems

By

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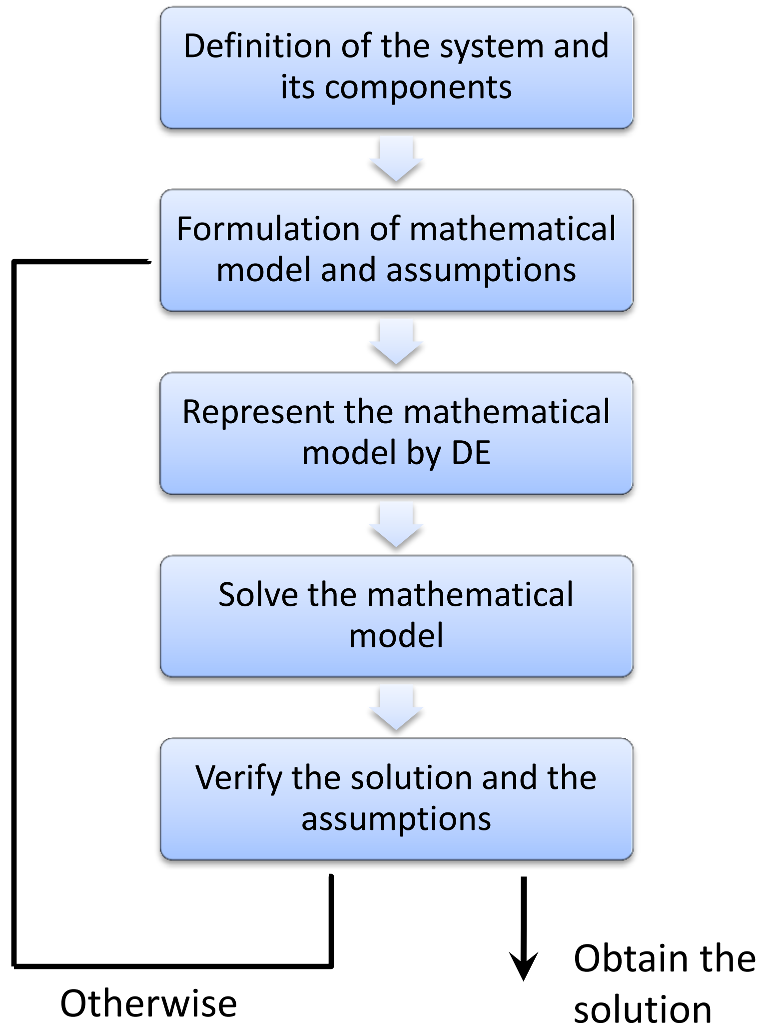
Mathematical Modeling of mechanical end electrical systems

Dynamic system modeling



Through and across – variables

A through – variable is the variable that does not change between the ends of system element. For example, a current passing through a resistance

Across – variable is the variable that changes between the ends of system element. For example, the voltage at the ends of a resistance

System	Through variable	Across variable
Electrical	Current, *i*	Voltage diff., *v*
Translational motion	Force, *F*	Velocity diff., *V*
Rotational motion	Torque, *T*	Angular velocity, ω
Fluids	Flow rate, *Q*	Pressure, *P*
Thermal	Heat flow, *q*	Temp. diff., *T*

DE for ideal elements

System	Governing DE	System	Governing DE
Elec. Inductance	$v = L\,(di/dt)$	Fluid caps.	$Q = Cf\,(dP/dt)$
Trans. Spring	$V = (1/k)(dF/dt)$	Thermal caps.	$q = Ct\,(dT_{emp}/dt)$
Rot. Spring	$\omega = (1/k)(dT/dt)$	Elec. Resistance	$i = (1/R)\,v$
Fluid inertia	$P = I\,(dQ/dt)$	Trans. Damper	$F = C\,v$
Elec. Capacitance	$i = C\,(dv/dt)$	Rot. Damper	$T = C\,\omega$
Mass	$F = M\,(dv/dt)$	Fluid resistance	$Q = (1/R_f)\,P$
Mass moment of inertia	$T = J\,(d\omega/dt)$	Thermal resistance	$Q = (1/R_t)T$

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Analogous systems

Analogous systems are systems that have the same governing differential equation

Example

Mass – spring damper system

$$M\frac{dV(t)}{dt}+DV(t)+k\int_0^t V(t).dt=F(t)$$

RLC circuit

$$C\frac{dv(t)}{dt}+\frac{v(t)}{R}+\frac{1}{L}\int_0^t v(t).dt=R(t)$$

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Laws for modeling

In control systems, the main systems are combination between mechanical and electrical systems. For mechanical systems, Newton's laws are used and in the electrical systems Kirchhoff's laws are used

Newton 2nd law of motion

$$\sum F = ma$$

Where:
F is the force
m is the mass
a in the acceleration

Kirchhoff's laws

Junction Rule $\sum I_i = \sum I_o$ **where *I* is the current**

Close Loop Rule $\sum \Delta V_{closed\,loop} = 0$ **where *V* is the voltage**

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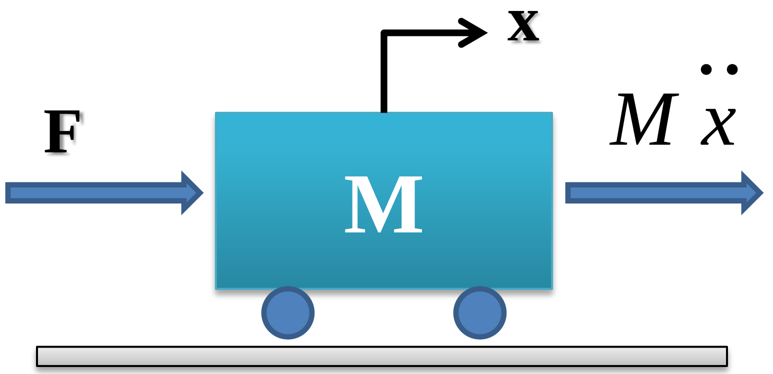
Newton 2nd law of motion

Statement

When a net external force acts on an object of mass *m, the acceleration that results is* directly proportional to the net force and has a magnitude that is inversely proportional to the mass. The direction of the acceleration is the same as the direction of the net force.



Law

$$\sum F = Ma = M\ddot{x}$$

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Kirchhoff's laws

Junction Rule Statement

- "At any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node, or: The algebraic sum of currents in a network of conductors meeting at a point is zero".
- The sum of currents entering the junction are thus equal to the sum of currents leaving. This implies that the current is conserved (no loss of current).

Law $\sum I_i = \sum I_o$

Close Loop Rule Statement

The principles of conservation of energy imply that the directed sum of the electrical potential differences (voltage) around any closed circuit is zero.

Law $\sum \Delta V_{closed\,loop} = 0$

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Linear approximation of systems

The physical system can be linear for a range and none linear if we extend this range. For example, the helical spring has a liner relation between the force applied on it and the deflection if the deflection was small.

Linear system has an excitation (x(t)) and response (y(t))

For example

$$M\frac{dV(t)}{dt} + DV(t) + k\int_{0}^{t} V(t).dt = F(t)$$

Response

Excitation

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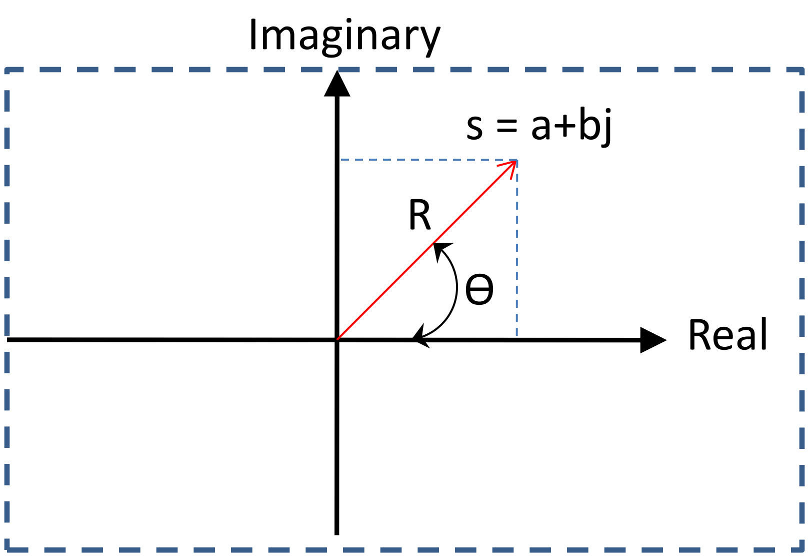
Complex variable

A complex variable is a combination of real and imaginary variables (a and b respectively)

$$s = a + bj$$

Complex variable (s) **Real** (a) **Imaginary** (b)



s-plane

$$a = R\cos(\theta) \qquad R = \sqrt{a^2 + b^2}$$

$$b = R\sin(\theta) \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

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Complex variable

A complex variable (s) is a combination of real variable (a) and imaginary variable (b) and is represented as s = a+bj

or it can be represented using Taylor series as: $e^{j\Theta} = R\cos(\Theta) + R\sin(\Theta)\,j$

A complex variable can be represented as : $s = R\cos(\Theta) + R\sin(\Theta)\,j$

Complex variable operations

Assume $s_1 = a_1+b_1j$ and $s_2=a_2+b_2j$, then

$s_1 \pm s_2 = (a_1+a_2) \pm (b_1+b_2)j$

$s_1 \pm s_2 = (R_1 \pm R_2)\, e^{(\Theta 1 \pm \Theta 2)j}$

$s_1 s_2 = (a_1a_2-b_1b_2) - (a_1b_2-b_1a_2)j$

$s_1 s_2 = (R_1R_2)\, e^{(\Theta 1 + \Theta 2)j}$

$s_1/s_2 = (R_1/R_2)\, e^{(\Theta 1 - \Theta 2)j}$

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Complex functions

A complex function G(s) is a function of complex variables of (s) kind

Example

$$G(s)=\frac{s^2+2s+10}{s^2\left(s^3+20s^2+3s+6\right)}$$

The values of (s) that make G(s) = 0 are called **zeros**

The values of (s) that make G(s) = ±∞ are called **poles**

Example

$$G(s)=\frac{(s-1)(s+4)}{s^2(s+6)(s+7)^3(s+10)}$$

This function has zeros: 1, -4 and ±∞ and poles: 0 (double), -6, -7 (triple) and 10

Laplace transformation

Definition

$$F(s)=\int_0^{\infty} f(t)e^{-st}.dt=\ell(f(t))$$

F(s) is the Laplace transformation for the function f(t)

Laplace transformation is used usually to transfer the DE from time (t) domain to s-domain to reduce its complexity and then solved the transfer function and represent the solution again in the time domain

Laplace transformation is used in control systems to find a relation between the output and the input. The procedure is simple, transform the DE to s-domain and represent the response and the excitation as functions in terms of s (e.g. Y(s) and R(s) respectively), then the transfer function (G(s)) will be Y(s)/R(s).

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Laplace transformation

Table

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{\left(s^2+a^2\right)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s\left(s^2+3a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$

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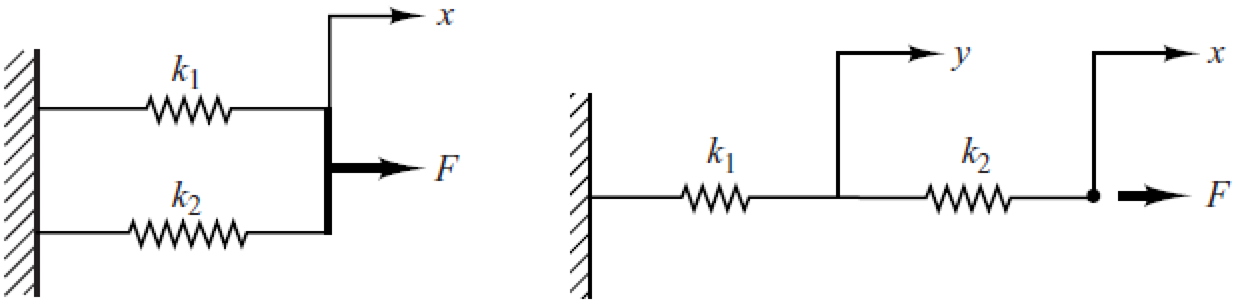
Laplace transformation

Table

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at}, \quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

A u t o m a t i c C o n t r o l

Mathematical modeling of springs

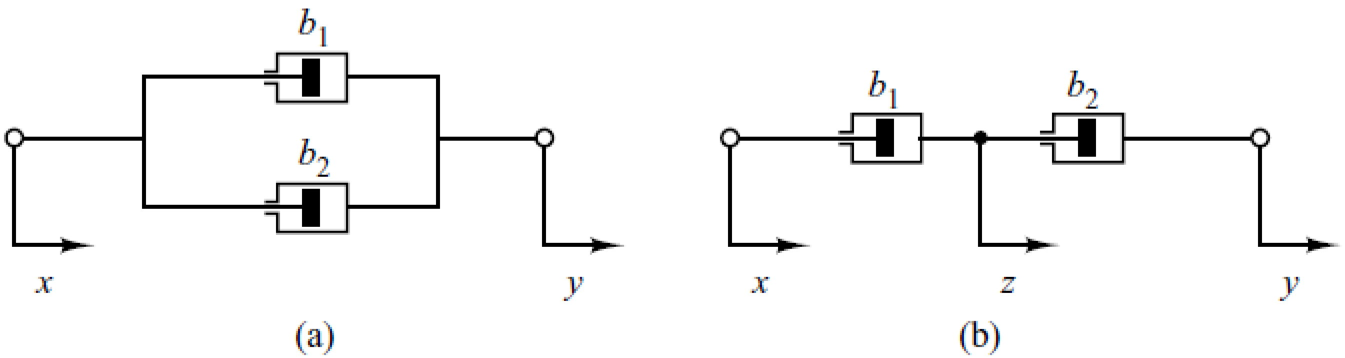


(a) (b)

$$k_{eq} = k_1 + k_2$$

$$k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Mathematical modeling of Damper



$$b_{eq} = b_1 + b_2$$

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$$

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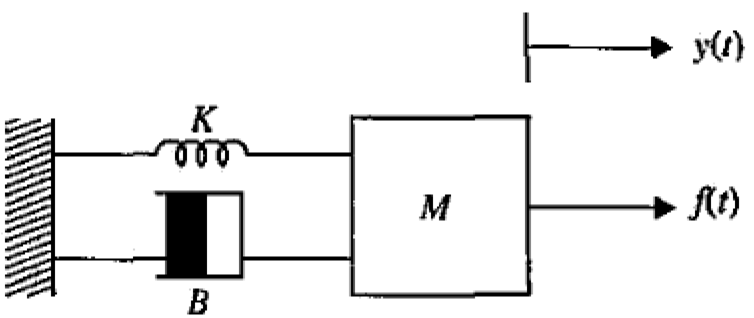
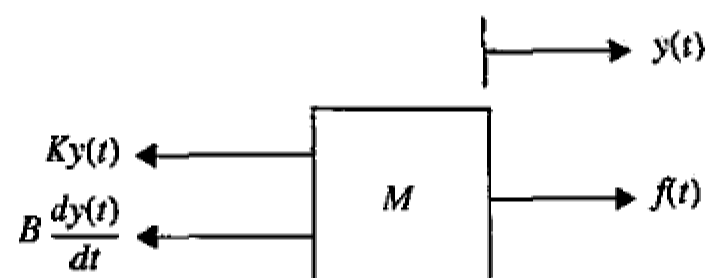
EXAMPLE: Consider the mass-spring-friction system shown in Fig. The linear motion concerned is in the horizontal direction. The free-body diagram of the system is shown in Fig. (b). The force equation of the system is

$$f(t) - B\frac{dy(t)}{dt} - Ky(t) = M\frac{d^2y(t)}{dt^2} \quad \Rightarrow \quad \frac{d^2y(t)}{dt^2} = -\frac{B}{M}\frac{dy(t)}{dt} - \frac{K}{M}y(t) + \frac{1}{M}f(t)$$

$$\ddot{y}(t) + \frac{B}{M}\dot{y}(t) + \frac{K}{M}y(t) = \frac{1}{M}f(t) \quad \Rightarrow \quad \frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$



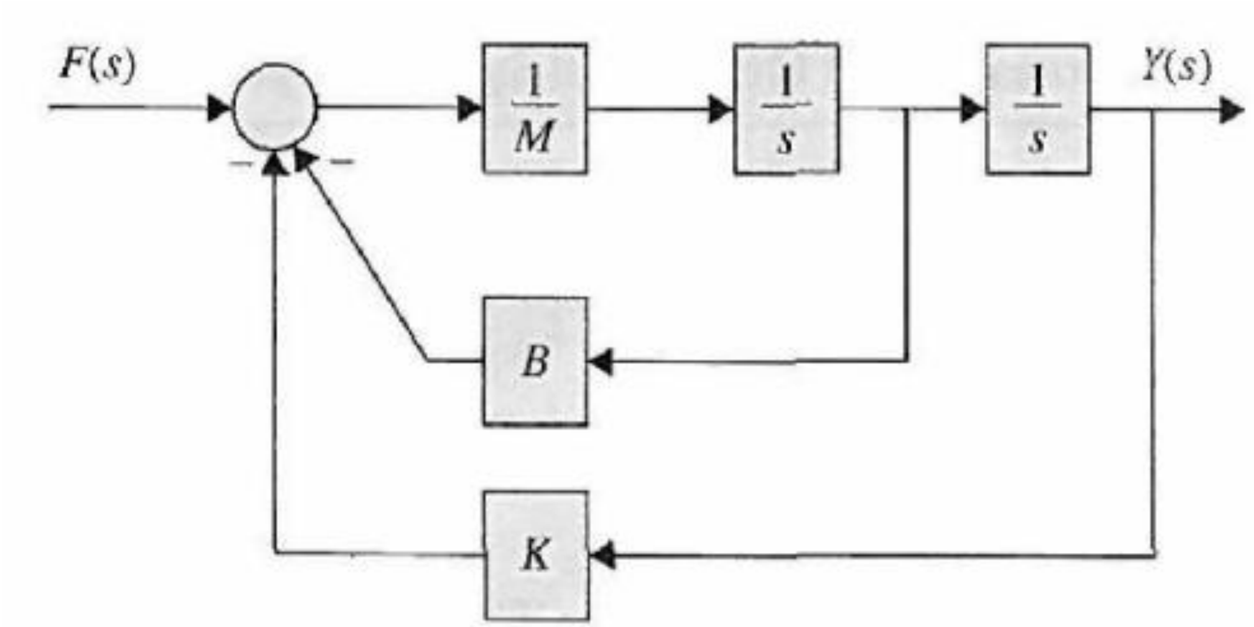
(a)



(b)

EXAMPLE:

Block diagram for mass-damper-spring system



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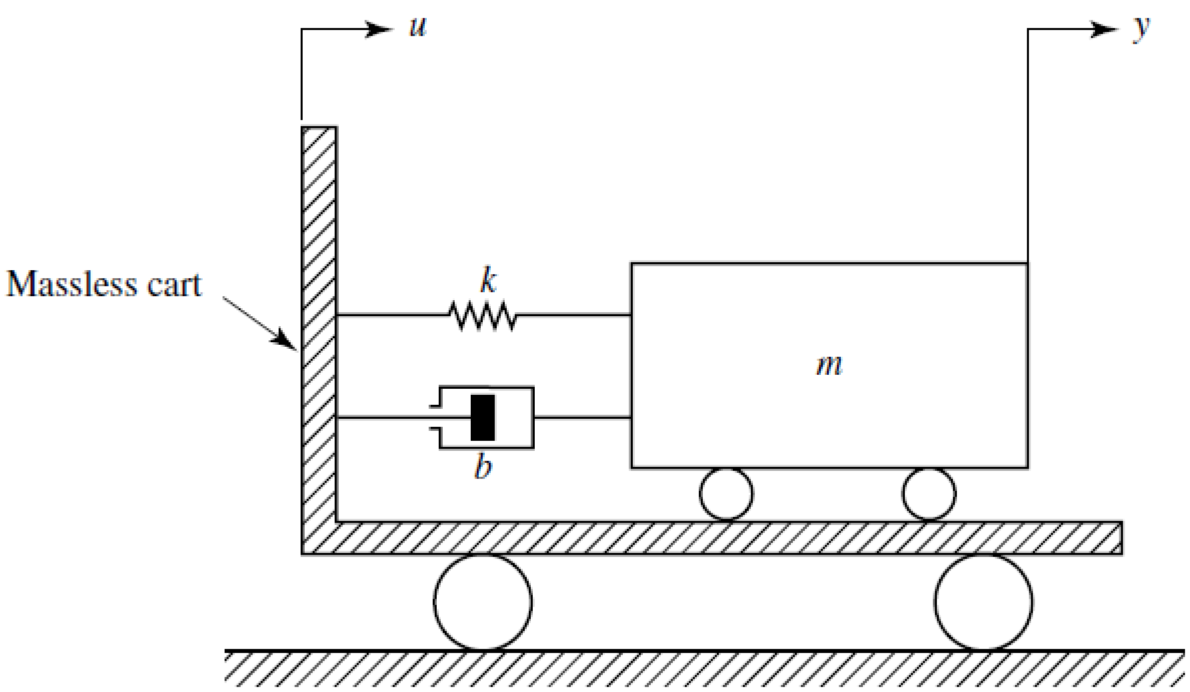
EXAMPLE 3–3

obtain mathematical models of this system by assuming that the cart is standing still for t<0 and the spring-mass-dashpot system on the cart is also standing still for t<0.



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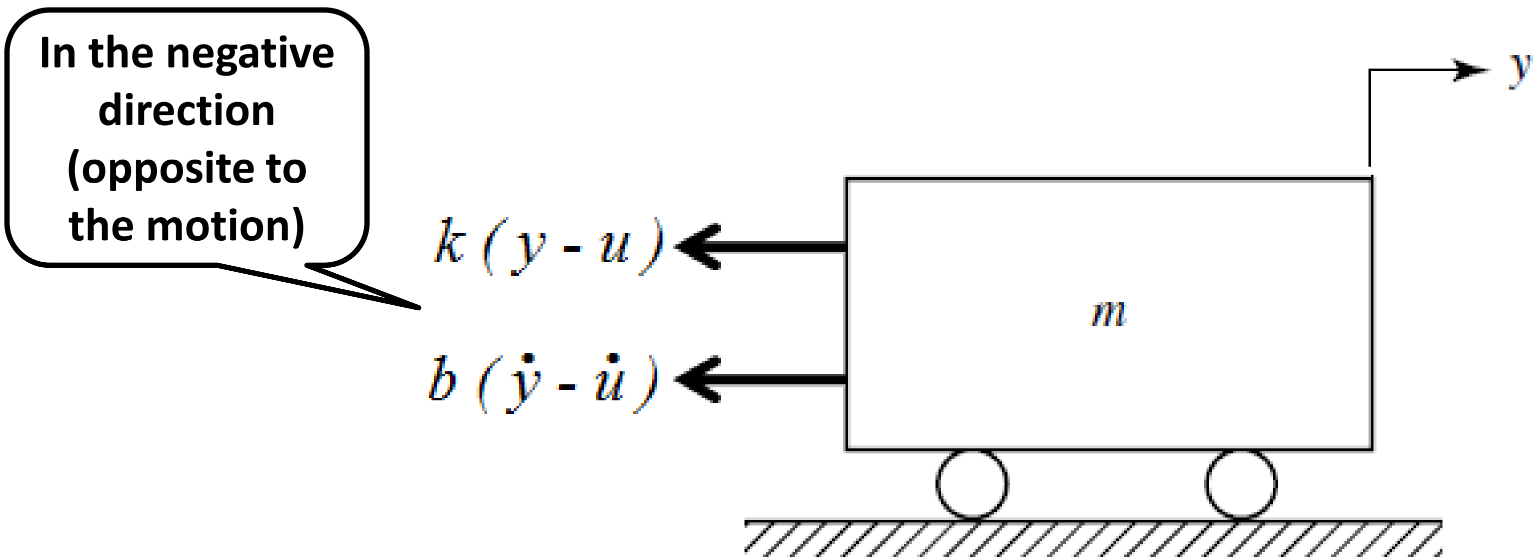
EXAMPLE 3–3

F.B.D



EXAMPLE 3–3

Solution

Apply Newton's 2nd law of motion $ma = \sum F$

You will obtain the following relation:- $m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$

Rearrange the terms:- $m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$ **. This equation is the mathematical of the system.**

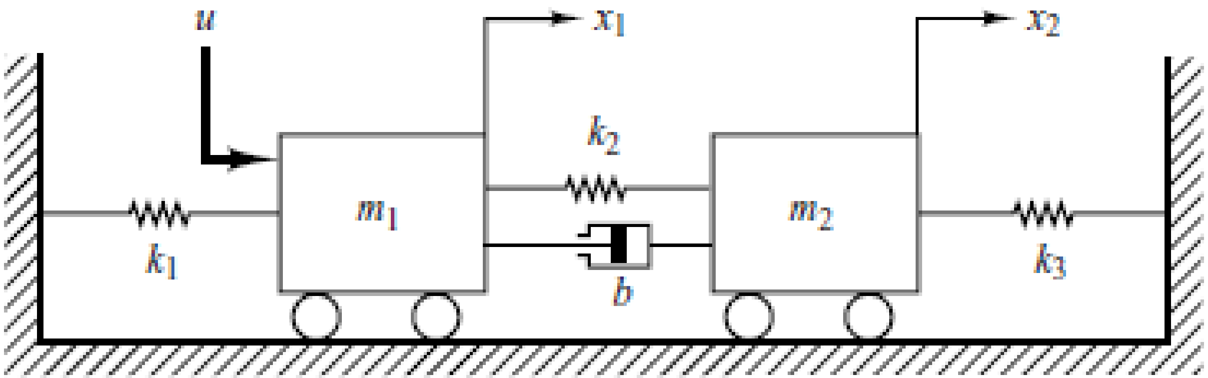
to find a transfer function, assume u is the input and y is the output. Take Laplace for the mathematical model. Assume zero I.Cs:-

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Rearrange this equation $\text{Transfer function} = G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$

EXAMPLE 3–4 Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–4.

Figure 3–4
Mechanical system.



EXAMPLE 3-4

Solution

The equations of motion for the system shown in Figure 3–4 are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Simplifying, we obtain

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$\left[m_1s^2 + bs + (k_1 + k_2)\right]X_1(s) = (bs + k_2)X_2(s) + U(s) \qquad (3\text{–}5)$$

$$\left[m_2s^2 + bs + (k_2 + k_3)\right]X_2(s) = (bs + k_2)X_1(s) \qquad (3\text{–}6)$$

EXAMPLE 3-4

Solution

Solving Equation (3–6) for $X_2(s)$ and substituting it into Equation (3–5) and simplifying, we get

$$\left[(m_1 s^2 + bs + k_1 + k_2)(m_2 s^2 + bs + k_2 + k_3) - (bs + k_2)^2\right] X_1(s)$$
$$= (m_2 s^2 + bs + k_2 + k_3) U(s)$$

from which we obtain

$$\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + bs + k_2 + k_3}{(m_1 s^2 + bs + k_1 + k_2)(m_2 s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \qquad (3\text{–}7)$$

From Equations (3–6) and (3–7) we have

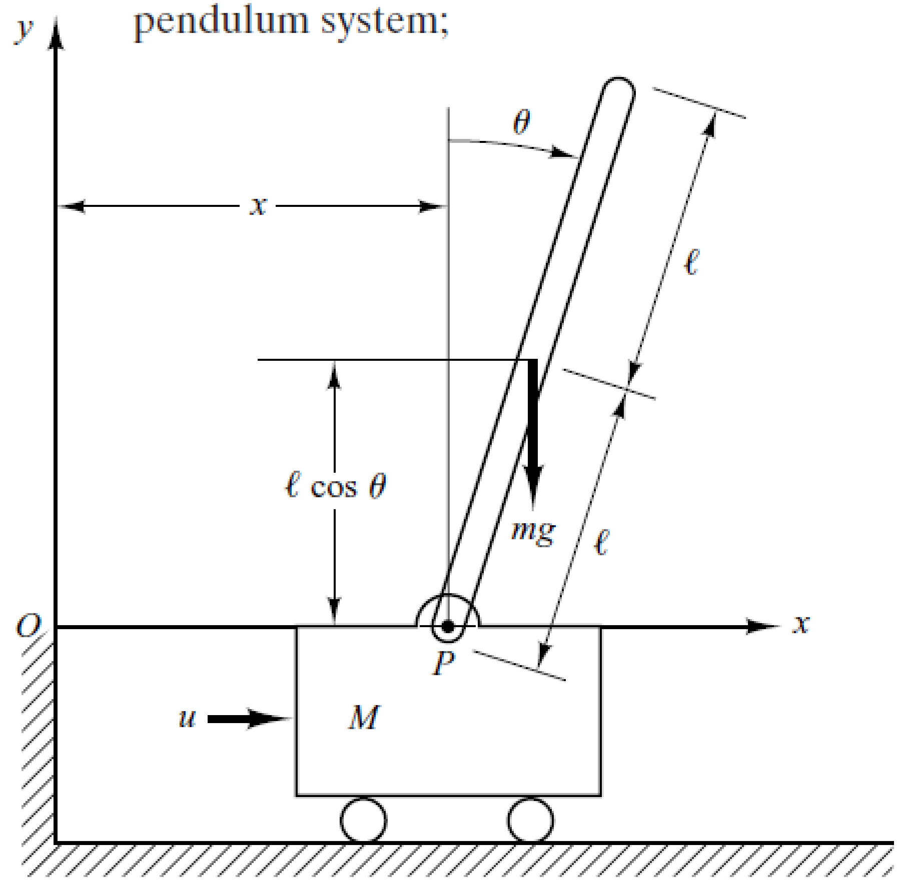
$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1 s^2 + bs + k_1 + k_2)(m_2 s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \qquad (3\text{–}8)$$

Equations (3–7) and (3–8) are the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$, respectively.

EXAMPLE 3-5

An inverted pendulum mounted on a motor-driven cart is shown in Figure 3–5(a).This is a model of the attitude control of a space booster on takeoff. (The objective of the attitude control problem is to keep the space booster in a vertical position.) The inverted pendulum is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider

Figure 3–5
(a) Inverted pendulum system;

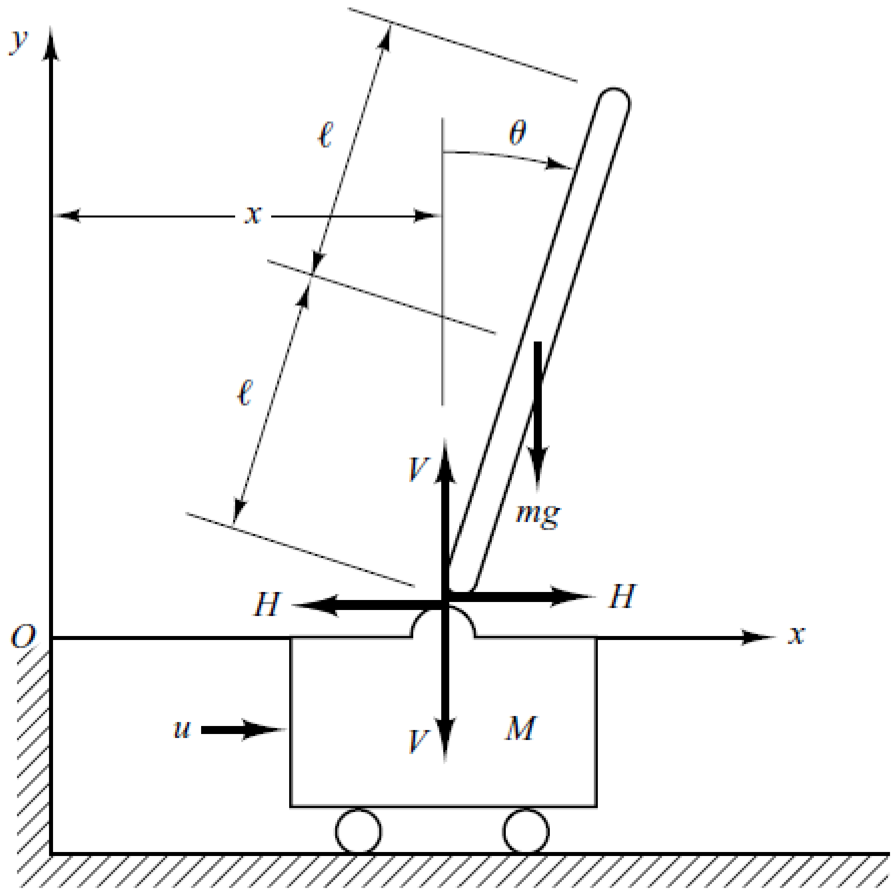


(a)

EXAMPLE 3-5

Figure 3–5

(b) free-body diagram.



(b)

EXAMPLE 3-5

only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force u is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center. Obtain a mathematical model for the system.

Define the angle of the rod from the vertical line as θ. Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G,y_G). Then:

$$x_G = x + l\sin\theta$$

$$y_G = l\cos\theta$$

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EXAMPLE 3-5

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 3–5(b). The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \tag{3–9}$$

where I is the moment of inertia of the rod about its center of gravity.
The horizontal motion of center of gravity of pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \tag{3–10}$$

The vertical motion of center of gravity of pendulum rod is

$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \tag{3–11}$$

EXAMPLE 3-5

The horizontal motion of cart is described by

$$M \frac{d^2 x}{dt^2} = u - H \qquad (3\text{–}12)$$

Since we must keep the inverted pendulum vertical, we can assume that $\theta(t)$ and $\dot{\theta}(t)$ are small quantities such that sin(θ)= θ, cos(θ) = 1 and $\theta\dot{\theta}^2 = 0$.
Then. Equations (3–9) through (3–11) can be linearized. The linearized equations are

$$I\ddot{\theta} = Vl\theta - Hl \qquad (3\text{–}13)$$

$$m(\ddot{x} + l\ddot{\theta}) = H \qquad (3\text{–}14)$$

$$0 = V - mg \qquad (3\text{–}15)$$

EXAMPLE 3-5

From Equations (3–12) and (3–14), we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–16}$$

From Equations (3–13), (3–14), and (3–15), we have

$$\begin{aligned} I\ddot{\theta} &= mgl\theta - Hl \\ &= mgl\theta - l(m\ddot{x} + ml\ddot{\theta}) \end{aligned}$$

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–17}$$

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EXAMPLE 3-6

Consider the inverted-pendulum system shown in Figure 3–6. Since in this system the mass is concentrated at the top of the rod, the center of gravity is the center of the pendulum ball. For this case, the moment of inertia of the pendulum about its center of gravity is small, and we assume $I = 0$ in Equation (3–17). Then the mathematical model for this system becomes as follows:

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \qquad (3\text{–}18)$$

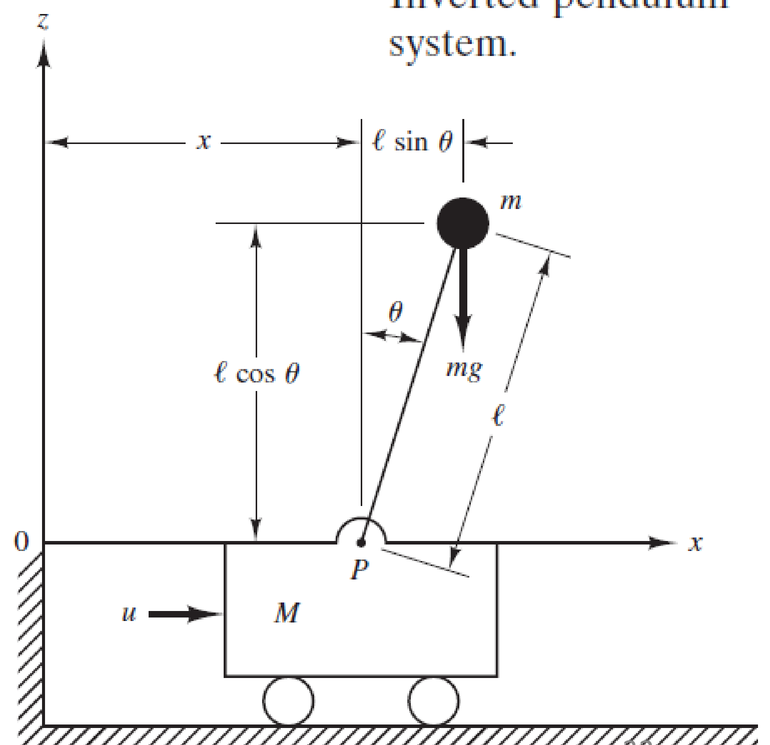
$$ml^2\ddot{\theta} + ml\ddot{x} = mgl\theta \qquad (3\text{–}19)$$

Equations (3–18) and (3–19) can be modified to

$$Ml\ddot{\theta} = (M + m)g\theta - u \qquad (3\text{–}20)$$

$$M\ddot{x} = u - mg\theta \qquad (3\text{–}21)$$

Figure 3–6 Inverted-pendulum system.



EXAMPLE 3-6

Solution

Equation (3–20) was obtained by eliminating $\ddot{x}$ from Equations (3–18) and (3–19). Equation (3–21) was obtained by eliminating $\ddot{\theta}$ from Equations (3–18) and (3–19). From Equation (3–20) we obtain the plant transfer function to be

$$\frac{\Theta(s)}{-U(s)} = \frac{1}{Mls^2 - (M + m)g}$$

$$= \frac{1}{Ml\left(s + \sqrt{\frac{M + m}{Ml}g}\right)\left(s - \sqrt{\frac{M + m}{Ml}g}\right)}$$

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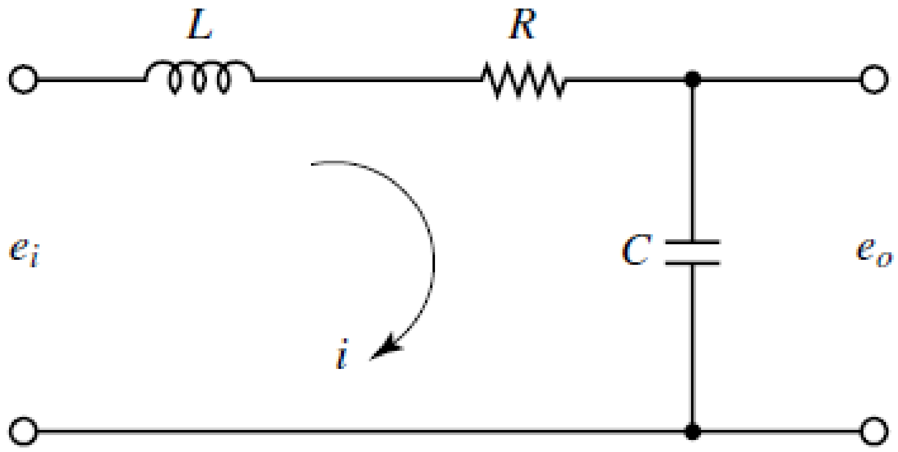
Mathematical modeling of electrical systems

LRC Circuit.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \qquad (3\text{–}24)$$

$$\frac{1}{C}\int i\,dt = e_o \qquad (3\text{–}25)$$



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LRC Circuit.

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

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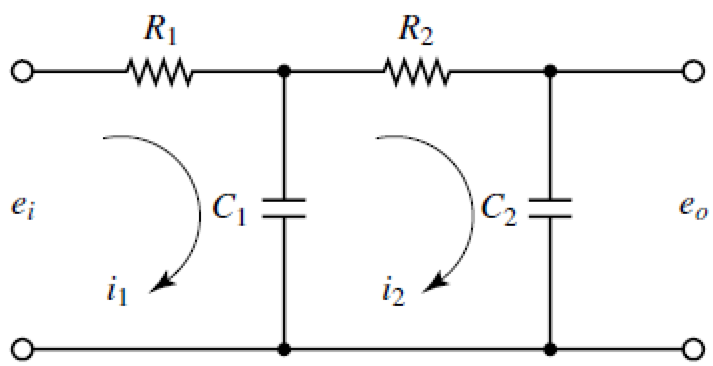
Transfer Functions of Cascaded Elements.



$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i \qquad (3\text{–}27)$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0 \qquad (3\text{–}28)$$

$$\frac{1}{C_2}\int i_2\,dt = e_o \qquad (3\text{–}29)$$

A u t o m a t i c C o n t r o l

Mathematical modeling of electrical systems

Transfer Functions of Cascaded Elements.

$$\frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right] + R_1 I_1(s) = E_i(s) \qquad (3\text{–}30)$$

$$\frac{1}{C_1 s}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \qquad (3\text{–}31)$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s) \qquad (3\text{–}32)$$

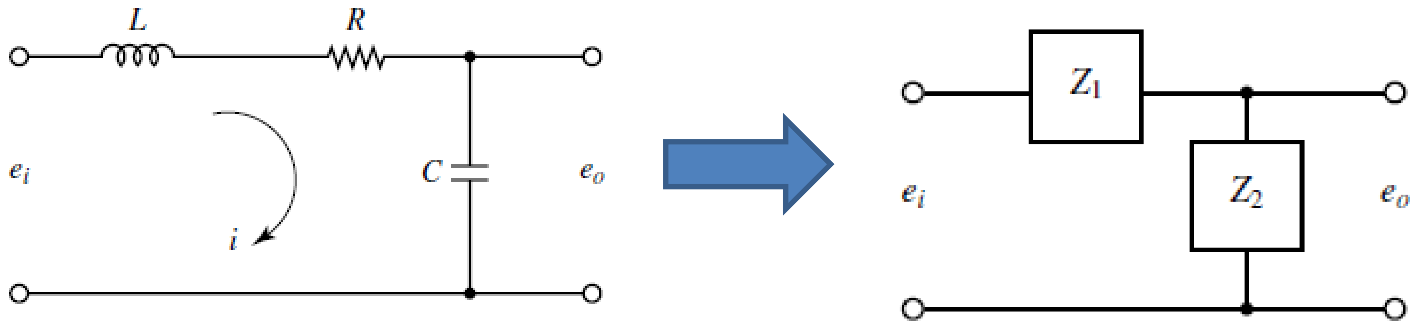
$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1} \qquad (3\text{–}33)$$

Mathematical Modeling of mechanical end electrical systems

Mathematical modeling of electrical systems

Transfer Functions of Cascaded Elements.



$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs} \quad \Rightarrow \quad \frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

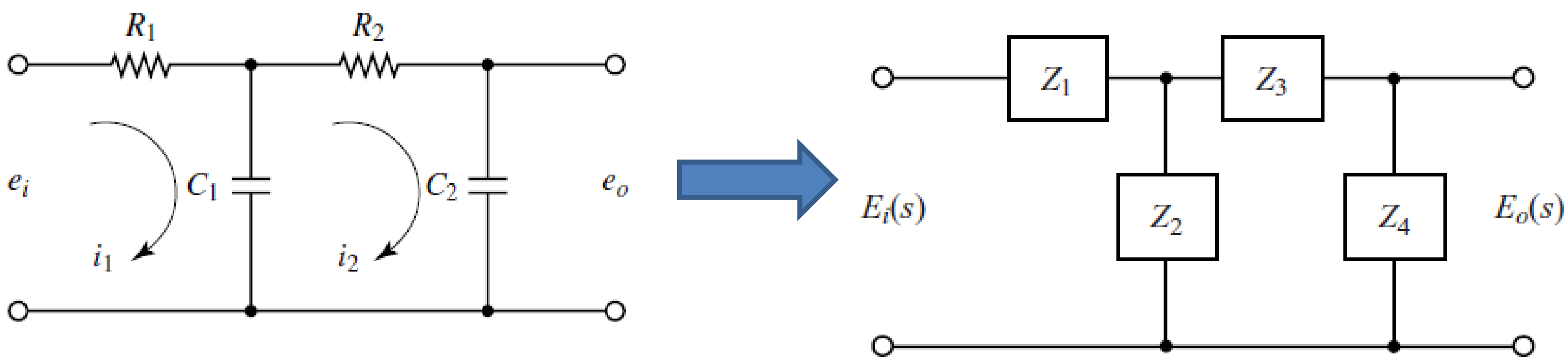
Remember that the impedance approach is valid only if the initial conditions involved are all zeros

Mathematical Modeling of mechanical end electrical systems

Mathematical modeling of electrical systems

Transfer Functions of Cascaded Elements.

EXAMPLE 3–7



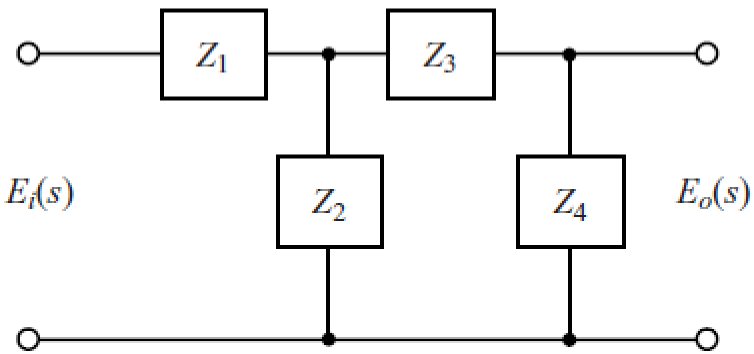
Where:- $Z_1 = R_1, Z_2 = 1/(C_1 s), Z_3 = R_2, \text{ and } Z_4 = 1/(C_2 s)$

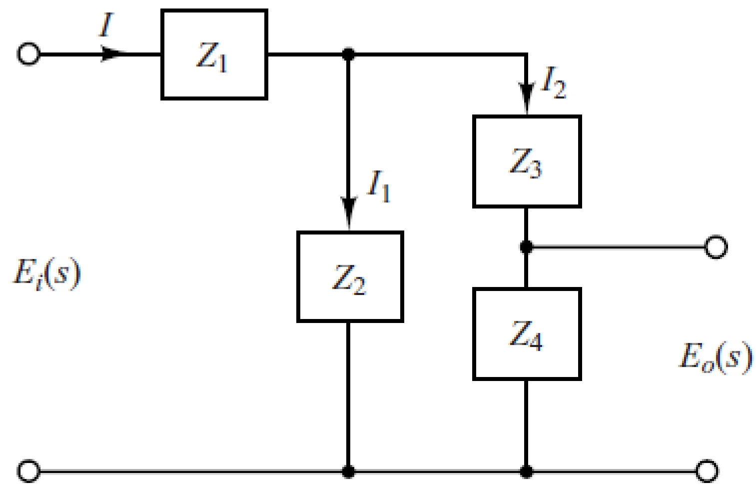
A u t o m a t i c C o n t r o l

Mathematical modeling of electrical systems

Transfer Functions of Cascaded Elements.

EXAMPLE 3–7





$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

A u t o m a t i c C o n t r o l

Mathematical modeling of electrical systems

Transfer Functions of Cascaded Elements.

EXAMPLE 3–7

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I \qquad E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)} = \frac{\frac{1}{C_1 s}\frac{1}{C_2 s}}{R_1\left(\frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}\right) + \frac{1}{C_1 s}\left(R_2 + \frac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

Mathematical modeling of electrical systems

Transfer Functions of Cascaded Elements.

EXAMPLE 3–7

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I \qquad E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

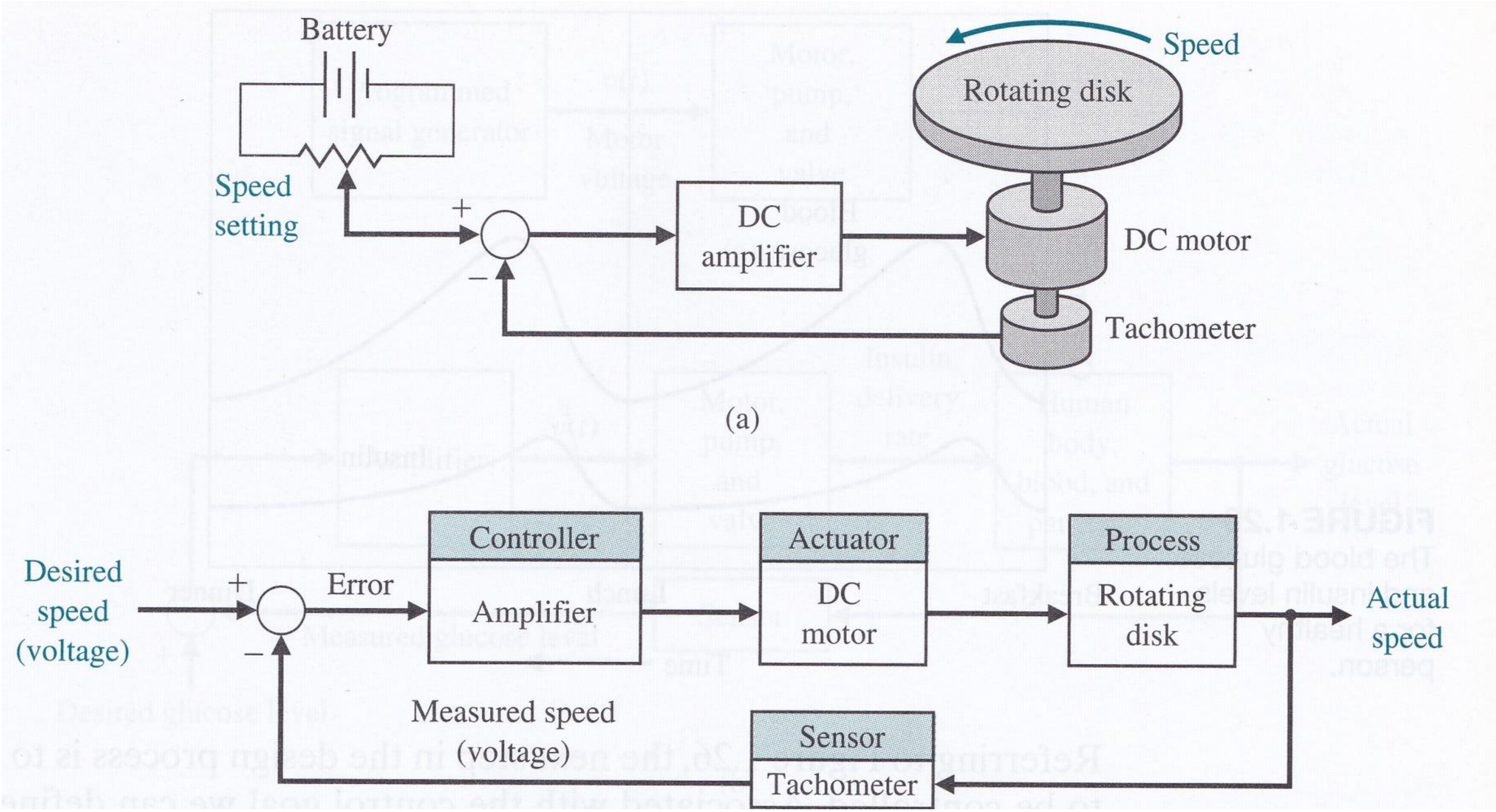
$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)} = \frac{\frac{1}{C_1 s}\frac{1}{C_2 s}}{R_1\left(\frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}\right) + \frac{1}{C_1 s}\left(R_2 + \frac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

Mathematical Modeling of mechanical end electrical systems

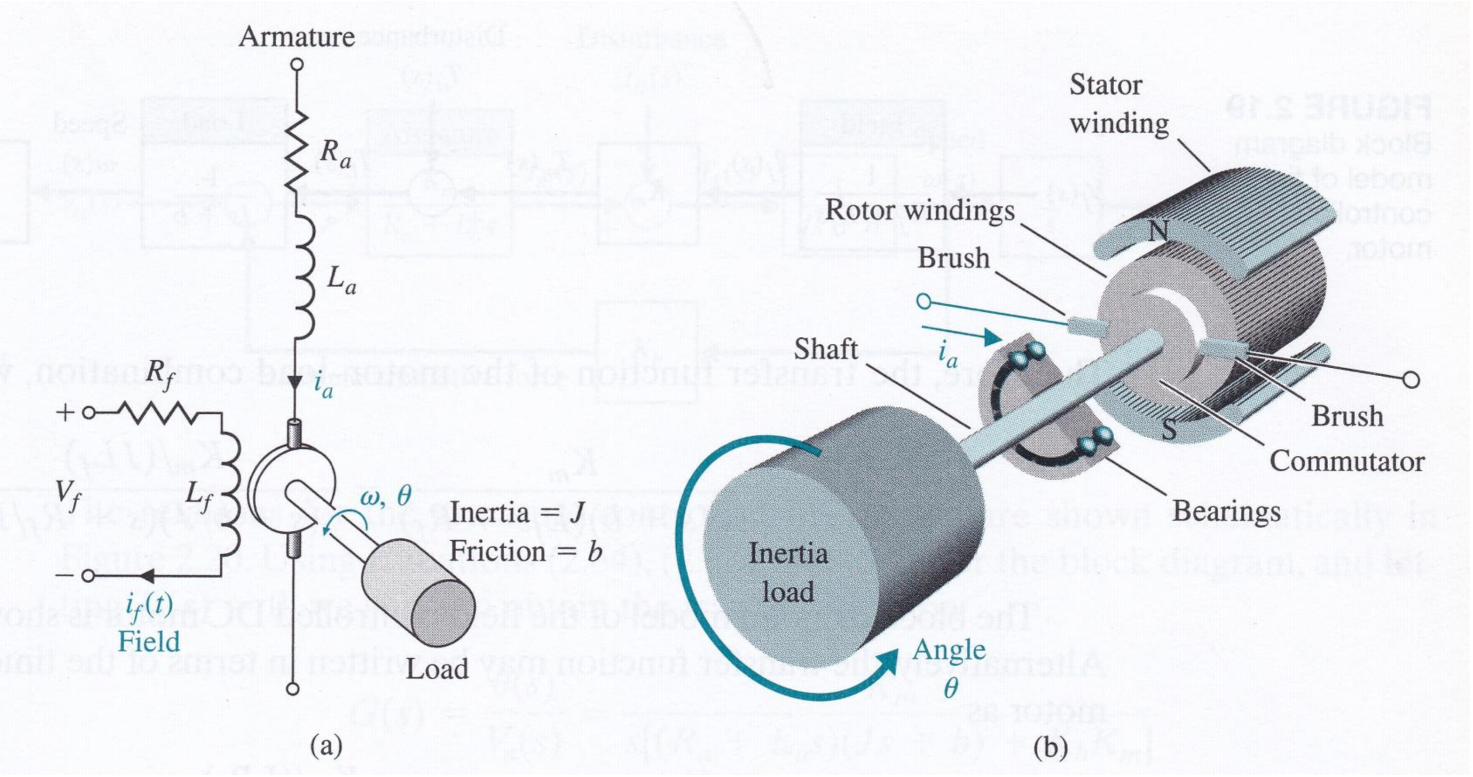
Mathematical modeling of electrical systems

DC motor



Mathematical modeling of electrical systems

DC motor



Mathematical Modeling of mechanical end electrical systems

Mathematical modeling of electrical systems

DC motor: field current control

$$T_m(s) = (K_1 K_f I_a) I_f(s) = K_m I_f(s),$$

$$T_m(s) = T_L(s) + T_d(s), \qquad T_L(s) = Js^2\theta(s) + bs\theta(s).$$

$$T_L(s) = T_m(s) - T_d(s), \quad T_m(s) = K_m I_f(s), \quad I_f(s) = \frac{V_f(s)}{R_f + L_f s}.$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)} = \frac{K_m/(JL_f)}{s(s + b/J)(s + R_f/L_f)}.$$

Mathematical Modeling of mechanical end electrical systems

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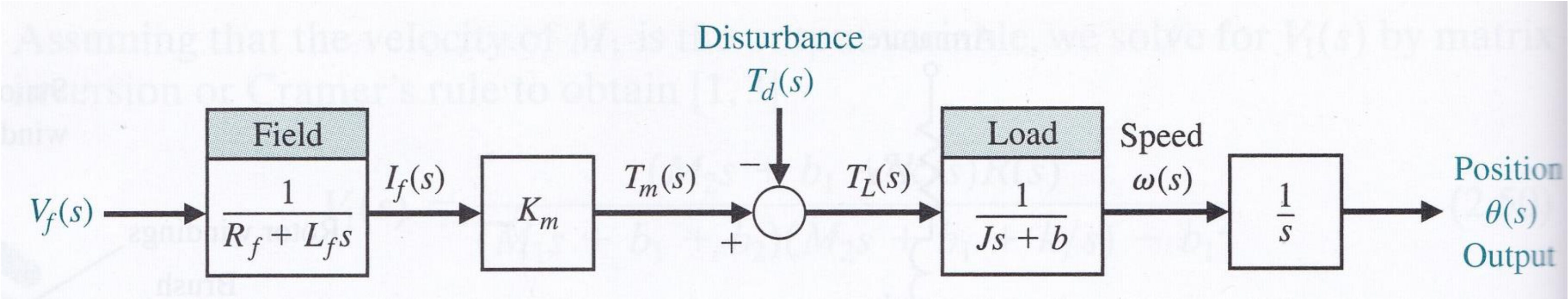
DC motor: field current control



A u t o m a t i c C o n t r o l

Mathematical Modeling of mechanical end electrical systems

Mathematical modeling of electrical systems

DC motor: armature current control

$$T_m(s) = (K_1 K_f I_f) I_a(s) = K_m I_a(s).$$

$$I_a(s) = \frac{V_a(s) - K_b \omega(s)}{R_a + L_a s}.$$

$$V_a(s) = (R_a + L_a s) I_a(s) + V_b(s),$$

$$V_b(s) = K_b \omega(s),$$

$$T_L(s) = J s^2 \theta(s) + b s \theta(s) = T_m(s) - T_d(s).$$

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(J s + b) + K_b K_m]} = \frac{K_m}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Mathematical modeling of electrical systems

DC motor: armature current control